On Boman's Theorem On Partial Regularity Of Mappings

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Abstract. Let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ and k be a positive integer. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map such that for each $(\xi, \eta) \in \Lambda$, the derivatives $D_\xi^j f(x) := \frac{d^j}{dt^j} f(x + t\xi)\Big|_{t=0}$, $j = 1, 2, ...k$, exist and are continuous. In order to conclude that any such map f is necessarily of class C^k it is necessary and sufficient that Λ be *not* contained in the zero-set of a nonzero homogenous polynomial $\Phi(\xi, \eta)$ which is linear in $\eta = (\eta_1, \eta_2, ..., \eta_m)$ and homogeneous of degree k in $\xi = (\xi_1, \xi_2, ..., \xi_n)$.

This generalizes a result of J. Boman for the case $k = 1$. The statement and the proof of a theorem of Boman for the case $k = \infty$ is also extended to include the Carleman classes $C\{M_k\}$ and the Beurling classes $C(M_k)$.([**4**])

A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ that is differentiable when restricted to arbitrary differentiable curves is not necessarily differentiable as a function of several variables [**12**]. Indeed, there are discontinuous functions $f : \mathbb{R}^n \to \mathbb{R}$ whose restrictions to arbitrary analytic arcs are analytic [**2**]. But a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ whose restriction to every line segment is real analytic is necessarily real analytic ([**13**]). In [**8**], [**9**] , [**10**] and [**11**] this result was extended by considering restrictions to algebraic curves and surfaces of functions belonging to more general classes of infinitely differentiable functions. It is also well known that a function $f : \mathbb{R}^n \to \mathbb{R}$ that is infinitely differentiable in each variable separately may be no better than measurable ([**7**]). In [**4**], the obverse problem is considered; for vector valued functions hypothesis is made on the source as well as the target space. In this note, Theorem 4 of [**4**] is generalized to C^k, $k \geq 1$, the class of functions that have continuous derivatives up to order k.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally bounded map. For $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, set

$$D_\xi \langle f, \eta \rangle (x) := \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \Big|_{t=0} \quad \text{in the sense of distributions,}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^m$. By the Leibniz Integral rule, we have

$$\frac{d}{dt} \int \langle f(x + t\xi), \eta \rangle \, dx = \int \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \, dx.$$

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Let $k, 1 \leq k < \infty$, be fixed. For $\xi \in \mathbb{R}^n$, denote by $C_\xi^k(\mathbb{R}^n)$ the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the derivatives $D_\xi^j f(x) := \frac{d^j}{dt^j} f(x + t\xi)\Big|_{t=0}$, $j = 1, 2, ...k$, exist and are continuous. Similarly, $C_\xi^\infty(\mathbb{R}^n) := \cap_{k=0}^\infty C_\xi^k(\mathbb{R}^n)$.

We are interested in finding the necessary and sufficient conditions on a subset $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ to have the following property:

$$\begin{aligned} \text{if } f \quad &: \quad \mathbb{R}^n \to \mathbb{R}^m \text{ is locally bounded} \\ \text{such that } \langle f, \eta \rangle \quad &\in \quad C_\xi^k(\mathbb{R}^n), \forall (\xi, \eta) \in \Lambda, \text{ then } f \in C^k(\mathbb{R}^n). \end{aligned}$$

The case $k = 1$ and $k = \infty$ was dealt in [**4**].

Let $\mathbb{Z}_+^n$ denote all n-tuples of nonnegative integers. For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}_+^n$, set $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.The set $\mathbb{Z}_+^n$ of multi-indices is assumed to be ordered lexicographically i.e. for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{Z}_+^n$, define $\alpha \prec \beta$ if there is $i, 1 \leq i \leq n$, such that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$.

Let $k_n = \binom{k+n-1}{k}$ denote the number of monomials of degree k in n variables.

Then for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} \int D_\xi \langle f, \eta \rangle (x) \varphi(x) dx &= \frac{d}{dt} \int \langle f(x + t\xi), \eta \rangle \varphi(x) dx \Big|_{t=0} \\ = \frac{d}{dt} \left\langle \int f(x) \varphi(x - t\xi) dx, \eta \right\rangle \Big|_{t=0} &= \left\langle \int f(x) \frac{d}{dt} \varphi(x - t\xi) dx, \eta \right\rangle \Big|_{t=0} \\ = -\sum_i \xi_i \left\langle \int f(x) \partial_i \varphi(x - t\xi) dx, \eta \right\rangle \Big|_{t=0} &= \sum_{i,j} \xi_i \eta_j \int \partial_i f_j(x) \varphi(x) dx. \end{aligned}$$

By iteration, we obtain the formula for higher-order distributional derivatives:

$$D_\xi^p \langle f, \eta \rangle (x) = \sum_{|\alpha|=p} \sum_{j=1}^m \xi^\alpha \eta_j \partial^\alpha f_j(x). \tag{1}$$

Let

$$\mathcal{B}_k := \left\{ \Phi(\xi, \eta) = \sum_{j=1}^m \sum_{|\alpha|=k} \varphi_{\alpha j} \xi^\alpha \eta_j : \varphi_{\alpha j} \in \mathbb{R}, \alpha \in \mathbb{Z}_+^n, j \in \mathbb{Z}_+ \right\}.$$

For any function $\Phi(\xi, \eta)$, set $\|\Phi\| := \max_{\|\xi\| \leq 1, \|\eta\| \leq 1} |\Phi(\xi, \eta)|$. For a subset $K \subset\subset \Lambda$, ($\subset\subset$ denotes the compact inclusion) put $\|\Phi\|_K := \max_{(\xi,\eta) \in K} |\Phi(\xi, \eta)|$.

Theorem 1. *Let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ be a subset and k be a positive integer. The following conditions are equivalent.*

(i) Λ is not *contained in an algebraic hypersurface defined by an element of $\mathcal{B}_k$ i.e.*

$$\Phi \in \mathcal{B}_k, \ \Phi|_\Lambda \equiv 0 \Rightarrow \Phi \equiv 0;$$

(ii) there exists a set consisting of $m \cdot k_n$ points

$$(\xi^*, \eta^*) = \left\{ \left(\xi^{(p)}, \eta^{(p)} \right) \in \Lambda, p = 1, 2, ..., mk_n \right\} \text{ such that } \det \Delta(\xi^*, \eta^*) \neq 0,$$

where

$$\Delta(\xi^*, \eta^*) := \left[\left(\xi^{(p)} \right)^\alpha \eta_j^{(p)} \right]_{|\alpha|=k, 1 \leq j \leq m, 1 \leq p \leq mk_n};$$

(iii) if $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally bounded and $\langle f, \eta \rangle \in C_\xi^k, \forall (\xi, \eta) \in \Lambda$, then $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$.

If any one of the above equivalent conditions is satisfied, then there exists a constant B depending only on Λ such that the following inequality holds for all locally bounded maps $f:\mathbb{R}^n\to\mathbb{R}^m$:

$$\max_{1\le j\le m}\max_{|\alpha|=k}|\partial^\alpha f_j(x)|\le B\cdot\sup_{(\xi,\eta)\in\Lambda}\left|D_\xi^k\left\langle f,\eta\right\rangle(x)\right|,\forall x\in\mathbb{R}^n. \tag{2}$$

Proof. We will prove $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(i)$.

$(i)\Rightarrow(ii)$.

Suppose $\det\Delta\left(\xi^*,\eta^*\right)=0$ for every set of mk_n elements $\left(\xi^*,\eta^*\right)=\left\{\left(\xi^{(p)},\eta^{(p)}\right)\right\}_{1\le p\le mk_n}$ in Λ. Fix one such set $\left(\xi^*,\eta^*\right)$ so that the rank $l:=\operatorname{rank}\Delta\left(\xi^*,\eta^*\right)$ is positive. Let $\Delta^{(l)}$ denote some $l\times l$ submatrix of $\Delta\left(\xi^*,\eta^*\right)$ such that the minor $\det\Delta^{(l)}$ is nonzero. Let $\Delta^{(l+1)}$ be a $(l+1)\times(l+1)$ submatrix of $\Delta\left(\xi^*,\eta^*\right)$ that contains $\Delta^{(l)}$ as a submatrix. Replace the point $\left(\xi^{(p_0)},\eta^{(p_0)}\right)$ in $\Delta^{(l+1)}$ which does not appear in $\Delta^{(l)}$ by variables $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^m$. By expanding $\Delta^{(l+1)}$ along the row where the replacement took place we obtain an element

$$\Phi(\xi,\eta)=\sum_{\alpha,j}\varphi_{\alpha j}\xi^\alpha\eta_j,$$

of $\mathcal{B}_k$ which is nonzero since one of its coefficients coincides with det $\Delta^{(l)}$ up to a sign.

Since $\Delta\left(\xi^*,\eta^*\right)$ has rank l, we find that $\Phi(\xi,\eta)=0$ for all $(\xi,\eta)\in\left(\xi^*,\eta^*\right)$. If $\Phi(\xi,\eta)=0$ for all $(\xi,\eta)\in\Lambda$, we are done. Otherwise, choose a point $(\widetilde{\xi},\widetilde{\eta})\in\Lambda\smallsetminus\left(\xi^*,\eta^*\right)$ with $\Phi(\widetilde{\xi},\widetilde{\eta})\neq 0$.

Let $\left(\widetilde{\xi^*},\widetilde{\eta^*}\right)$ be the set which is obtained from $\left(\xi^*,\eta^*\right)$ by replacing the point $\left(\xi^{(p_0)},\eta^{(p_0)}\right)$ by $(\widetilde{\xi},\widetilde{\eta})$. Then, the $\operatorname{rank}\Delta\left(\widetilde{\xi^*},\widetilde{\eta^*}\right)\ge l+1$. By repeating above procedure, we find a sequence of subsets $\left(\xi^*,\eta^*\right)^{(i)}\subset\Lambda$, $i=1,2,3,..$, each with mk_n elements such that the $\operatorname{rank}\Delta\left(\xi^*,\eta^*\right)^{(j)}$ is a strictly increasing sequence of nonnegative integers. After finitely many steps we obtain a nonzero element of $\mathcal{B}_k$ which vanishes on the entire Λ.

$(ii)\Rightarrow(iii)$.

Let $\left(\xi^*,\eta^*\right)=\left\{\left(\xi^{(p)},\eta^{(p)}\right)\in\Lambda\right\}_{1\le p\le mk_n}$ be a set of points such that $\det\Delta\left(\xi^*,\eta^*\right)\neq0$. By applying Cramer's rule to (1), we get

$$\partial^\alpha f_j(x)=\sum_{p=1}^{mk_n}\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}D_{\xi^{(p)}}^k\left\langle f,\eta^{(p)}\right\rangle(x)\text{ in the distributional sense,}$$

where $\Delta_{\alpha j}^{(p)}$ denotes the cofactor obtained by deleting the (α,j)-th row and the p-th column. Since $D_\xi^k\left\langle f,\eta\right\rangle\in C^0$ for all $(\xi,\eta)\in\Lambda$, we have

$$\partial^\alpha f_j(x)=\sum_{p=1}^{mk_n}\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}D_{\xi^{(p)}}^k\left\langle f,\eta^{(p)}\right\rangle(x)\in C^0.$$

Furthermore, there exists a constant $B=B(k,f,\Lambda)$ such that

$$|\partial^\alpha f_j(x)|\le\sum_{p=1}^{mk_n}\left|\frac{\det\Delta_{\alpha j}^{(p)}}{\det\Delta}\right|\left|D_{\xi^{(p)}}^k\left\langle f,\eta^{(p)}\right\rangle(x)\right|\le B\cdot\sup_{(\xi,\eta)\in\Lambda}\left|D_\xi^k\left\langle f,\eta\right\rangle(x)\right|,$$

for all α with $|\alpha| = k$, and all $j = 1, 2, ..., m$.

$(iii) \Rightarrow (i)$.

Suppose (i) does not hold. Let $\Phi \in \mathcal{B}_k$ be such that $\Phi|_\Lambda \equiv 0$. We can write $\Phi(\xi, \eta) = \langle \varphi.(\xi), \eta \rangle$, where $\varphi.(\xi) := (\varphi_1(\xi), \varphi_2(\xi), ..., \varphi_m(\xi))$ and $\varphi_j(\xi) = \sum_{|\alpha|=k} \varphi_{\alpha j} \xi^\alpha$, $j = 1, 2, ..., m$, homogeneous polynomials of degree k.

Define the map

$$f(x) := \begin{cases} (\ln|\ln|x||)\, \varphi.(x) \text{ if } x \neq 0, \\ 0 \text{ if } x = 0 \end{cases}.$$

Clearly $f \notin C^k$ and f is C^∞ in $\{x \in \mathbb{R}^n : 0 < |x| < 1\}$. We will prove that $D_\xi^k \langle f(x), \eta \rangle$ exists at $x = 0$, for all $(\xi, \eta) \in \Lambda$. It is easy to see that here are constants C_α such that

$$|\partial^\alpha \ln|\ln|x||| \leq \frac{C_\alpha}{|x|^{|\alpha|} |\ln|x||}, \forall \alpha, |\alpha| \geq 1.$$

Since the $\varphi_j(x)$'s are homogeneous polynomials of degree k, when the Leibniz's formula is applied to the products $(\ln|\ln|x||)\, \varphi_j(x)$, it is clear that all terms in $D_\xi^p \langle f(x), \eta \rangle$, $1 \leq p \leq k$, except possibly

$$(\ln|\ln|x||) \left\langle D_\xi^k \varphi.(x), \eta \right\rangle \tag{3}$$

tend to 0 as $x \to 0$. We only need to prove that the function in (3) also tends to 0 as $x \to 0$. By expanding $(x_1 + t\xi_1)^{\alpha_1} (x_2 + t\xi_2)^{\alpha_2} ... (x_n + t\xi_n)^{\alpha_n}$ binomially, we can write

$$\varphi.(x + t\xi) := \varphi.(x) + P(x, \xi, t) + \varphi.(\xi) t^k.$$

But since $(\xi, \eta) \in \Lambda$,

$$\left\langle D_\xi^k \varphi.(x), \eta \right\rangle = k! \langle \varphi.(\xi), \eta \rangle = 0.$$

It follows that $\left| D_\xi^p \langle f(0), \eta \rangle \right| = 0$ for $p \leq k$. Thus, $f \in C_\xi^k$ for all $(\xi, \eta) \in \Lambda$, but $f \notin C^k$. □

Remark 1. *(cf. [**6**]) Suppose (i) is satisfied for all $k \geq 0$. It would be of interest to know whether there exists a constant $\rho = \rho(\Lambda)$, depending only on some approperiate notion of capacity of Λ, so that (2) is satisfied with $B = (\rho(\Lambda))^{-k}$ for all f and and all k.*

Remark 2. *Suppose Λ satisfies (i) or (ii). The proof of Theorem 1 shows that if f is continuous and $D_\xi^k \langle f, \eta \rangle = 0, \forall (\xi, \eta) \in \Lambda$, then f is a polynomial. The assumption of continuity of f is not necessary but our proof is valid only if f is continuous. See [**4**].*

Remark 3. *If Λ satisfies (i), then Λ contains at least mk_n elements. Furthermore, if (i) holds for k then (i) also holds for all $j \leq k$. Suppose there exists $\Phi \in B_j, j < k$ such that $\Phi|_\Lambda \equiv 0$ but $\Phi \not\equiv 0$. Then, $\xi_1^{k-j} \Phi \in \mathcal{B}_k$, $\left. \xi_1^{k-j} \Phi \right|_\Lambda \equiv 0$ but this is a contradiction.*

Let $\{M_k\}_{k=0}^\infty$, be a sequence of nonnegative numbers. For $h > 0$ and $K \subset\subset \mathbb{R}^n$ define the seminorm on $C^\infty(\mathbb{R}^n)$,

$$p_{h,K}(f) = \sup_{\alpha \in \mathbb{Z}_+^n} \sup_{x \in K} \frac{|\partial^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

The spaces

$$C\{M_k\} = \{f \in C^\infty(\mathbb{R}^n) : \forall K \subset\subset \mathbb{R}^n, \exists h > 0, \text{ s.t. } p_{h,K}(f) < \infty\}$$

and

$$C(M_k) = \{f \in C^\infty(\mathbb{R}^n) : p_{h,K}(f) < \infty, \forall K \subset\subset \mathbb{R}^n, \forall h > 0\}$$

are called the Carleman and Beurling classes, respectively. The classes $C\{(k!)^\nu\}$, $\nu > 1$, known as Gevrey classes, are especially important in partial differential equations and harmonic analysis. The class $C\{k!\}$ is precisely the class of real analytic functions.

We assume that

$$M_0 = 1 \text{ and } M_k \geq k!, \forall k; \tag{4}$$

$$M_k^{1/k} \text{ is strictly increasing;} \tag{5}$$

$$\exists C > 0 \text{ such that } M_{k+1} \leq C^k M_k, \ \forall k. \tag{6}$$

These conditions insure that the classes $C\{M_k\}$ and $C(M_k)$ are nontrivial and are closed under product and differentiation of functions. For more properties of these spaces, see [**5**], [**11**] and references there.

It is well known that $f \in C^\infty(\mathbb{R}^n)$ if and only if $\sup_{\xi\in\mathbb{R}^n} |\xi|^j |\widehat{\chi f}(\xi)| < \infty, \forall \chi \in C_c^\infty(\mathbb{R}^n), j \geq 1$. A similar characterization is also available for $C\{M_k\}$ (see [**5**]) a routine modification of which yields an analogous characterization of $C(M_k)$.

Let $r > 0$. Choose a sequence of cut-off functions $\chi_{(j)} \in C_c^\infty, j = 1, 2, ...,$ such that $\chi_{(j)}(x) = 1$ if $|x - x_0| < r$, $\chi_{(j)}(x) = 0$ if $|x - x_0| > 3r$ and

$$\left|\partial^\alpha \chi_{(j)}(x)\right| \leq (C_1 j)^{|\alpha|}, \forall j, \forall |\alpha| \leq j, \forall x,$$

where the constant C_1 is independent of j.

Then $f \in C\{M_k\}$ (*resp.* $C(M_k)$) in a neighborhood of $x_0 \in \mathbb{R}^n$ if and only if there exists a constant $\hbar > 0$ (*resp.* for every $\hbar > 0$)

$$\sup_{\xi\in\mathbb{R}^n} \sup_{j\geq 1} \hbar^{-j} M_j^{-1} |\xi|^j |\widehat{f\chi_{(j)}}(\xi)| < \infty.$$

Call a subset $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$ a determining set for bilinear forms of rank 1 if there is no nonzero bilinear form $\varphi(\xi, \eta), \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m$ of rank 1 such that $\varphi(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$.

Clearly Λ is a determining set for bilinear forms of rank 1 if and only if

$$\langle u, \xi\rangle \langle v, \eta\rangle = 0, \forall(\xi, \eta) \in \Lambda \Rightarrow |u||v| = 0$$

(here $\langle u, \xi\rangle$ and $\langle v, \eta\rangle$ are dot products on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively), or equivalently,

$$\cap_{(\xi,\eta)\in\Lambda}\{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m :< u, \xi >< v, \eta >= 0\} = (\mathbb{R}^n \times 0) \cup (0 \times \mathbb{R}^m).$$

Since $\mathbb{R}[u, v]$ is a Noetherian ring, Λ contains a finite subset Λ' such that the sets $\{< u, \xi >< v, \eta >: (\xi, \eta) \in \Lambda\}$ and $\{< u, \xi >< v, \eta >: (\xi, \eta) \in \Lambda'\}$ generate the same ideal in $\mathbb{R}[u, v]$ and thus define the same varieties:

$$\begin{aligned} \cap_{(\xi,\eta)\in\Lambda}\{(u, v) &\in \mathbb{R}^n \times \mathbb{R}^m :< u, \xi >< v, \eta >= 0\} \\ &= \cap_{(\xi,\eta)\in\Lambda'}\{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m :< u, \xi >< v, \eta >= 0\}. \end{aligned}$$

Thus, any determining set for bilinear forms of rank 1 contains a finite determining set for bilinear forms of rank 1.

Let $C\{M_k\}(\xi)$ (*resp.* $C(M_k)(\xi)$) denote the set of all $f \in C^{\infty}_{\xi}(\mathbb{R}^n)$ such that for every subset $K \subset\subset \mathbb{R}^n$, $\sup_{j,x\in K}\left|D^j_{\xi}f(x)\right|\hbar^{-j}M_j^{-1} < \infty, \forall j$, for some $\hbar > 0$ (*resp.* for every $\hbar > 0$).

Theorem 2. *Let $\{M_k\}_{k=0}^{\infty}$ be a sequence of nonnegative numbers satisfying the conditions (4), (5) and (6). The following statements are equivalent.*

(i) Λ is a determining set for bilinear forms of rank 1;

(ii) for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$\langle \eta, f\rangle \in C\{M_k\}(\xi), \forall(\eta,\xi) \in \Lambda \Rightarrow f \in C\{M_k\};$$

(iii) for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$\langle \eta, f\rangle \in C(M_k)(\xi), \forall(\eta,\xi) \in \Lambda \Rightarrow f \in C(M_k);$$

(iv) for any locally bounded map $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$\langle \eta, f\rangle \in C^{\infty}(\xi), \forall(\eta,\xi) \in \Lambda \Rightarrow f \in C^{\infty}.$$

Proof. (cf. Theorem 4 in [**4**]) Assume (i) holds. By the remark above, by replacing Λ by a subset, if necessary, we may assume Λ is finite. Suppose for every $(\eta,\xi) \in \Lambda$, $\langle \eta, f\rangle \in C\{M_k\}(\xi)$ (resp. $\langle \eta, f\rangle \in C(M_k)(\xi)$). Now for a suitable function f,

$$\langle \xi, z\rangle \widehat{\langle \eta, f\rangle}(z) = \langle \xi, z\rangle \left\langle \eta, \widehat{f}(z)\right\rangle = \left\langle \eta, i\int \left[\langle \xi, \partial_x\rangle e^{-i\langle x,z\rangle}\right] f(x)dx\right\rangle$$

$$= \left\langle \eta, -i\int e^{-i\langle x,z\rangle}\langle \xi, \partial_x f\rangle(x)dx\right\rangle = \left\langle \eta, -i\int e^{-i\langle x,z\rangle}D_{\xi}f(x)dx\right\rangle.$$

Let $g_{(j)} := f\chi_{(j)} \in C\{M_k\}$ near a fixed point x_0. Assume, without loss of generality, $x_0 = 0$. By assumption, for all $(\xi,\eta) \in \Lambda$ there exist constants $C = C_{\xi\eta}$ and $\hbar = \hbar_{\xi\eta} > 0$ (*resp.* for all $(\xi,\eta) \in \Lambda$ and for all $\hbar > 0$ there exists a constant $C = C_{\xi\eta,\hbar}$) such that

$$\left|\widehat{\langle \eta, g_{(j)}\rangle}(\zeta)\right| |\langle \xi, \zeta\rangle|^j = \left|\left\langle \eta, \widehat{g_{(j)}}(\zeta)\right\rangle\right| |\langle \xi, \zeta\rangle|^j \le C\hbar^j M_j, \forall(\xi,\eta) \in \Lambda, \zeta \in \mathbb{R}^n, j \in \mathbb{Z}_+.$$

The function

$$\mathbb{R}^n \times \mathbb{R}^m \ni (u,v) \to \sum_{(\xi,\eta)\in\Lambda} |\langle \eta, v\rangle| \, |\langle \xi, u\rangle|^l, \tag{7}$$

is homogeneous of degree 1 in v, of homogeneous degree l in u. Since none of the terms $|\langle \eta, v\rangle| \, |\langle \xi, u\rangle|$ can vanish on all of Λ, the function in (7) has a positive minimum on the compact set $\{(u,v) : |u| = 1, |v| = 1\}$. Thus, there is an $\varepsilon > 0$ such that

$$\sum_{(\xi,\eta)\in\Lambda} |\langle \eta, v\rangle| \, |\langle \xi, u\rangle|^l \ge \varepsilon |v||u|^l,$$

(see Lemma 1 [**4**]). Applying this to $u = \zeta, v = \widehat{g_{(j)}}(\zeta)$, we get

$$\left|\widehat{g_{(j)}}(\zeta)\right| |\zeta|^l \le \varepsilon^{-1}\sum_{(\xi,\eta)\in\Lambda}\left|\left\langle \eta, \widehat{g_{(j)}}(\zeta)\right\rangle\right| |\langle \xi, \zeta\rangle|^l \le C\hbar^j M_j,$$

where $\hbar = \max_{(\xi,\eta)\in\Lambda} \hbar_{\xi\eta}$ (*resp.* for all $\hbar > 0$) and $C = \varepsilon^{-1} \sum_{(\xi,\eta)\in\Lambda} C_{\xi\eta}$. Thus (ii) and (iii) hold. By setting $\hbar = 1$ and $M_j = 1, \forall j$, in the above argument, it is clear that (iii) holds as well.

Conversely if Λ is not a determinant set for bilinear forms of rank 1, there exist $u \neq 0$ and $v \neq 0$ such that

$$\langle u, \xi\rangle \langle v, \eta\rangle = 0, \forall (\xi, \eta) \in \Lambda.$$

Let $h : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous function. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined as $f(z) = h(\langle u, z\rangle) \cdot v$. Then

$$\left.\left(\frac{d}{dt}\langle \eta, f(z + t\xi)\rangle\right)\right|_{t=0} = \langle \eta, v\rangle \langle u, \xi\rangle h'(\langle u, z + t\xi\rangle)\big|_{t=0} \equiv 0$$

Thus $\langle \eta, f\rangle \in C(M_k)(\xi) \subset C\{M_k\}(\xi) \subset C^\infty(\xi), \forall(\xi, \eta) \in \Lambda$ but f need not be even differentiable. $\square$

References

[1] Agbor, D., and Boman, J., On modulus of continuity of mappings between Euclidean spaces, Mathematica Scandinavica. To appear

[2] Bierstone, E.; Milman, P. D.; Parusinski, A.: A function which is arc-analytic but not continuous. Proc. Amer. Math. Soc. 113 (1991) 419–423.

[3] Bochnak, J.: Analytic functions in Banach Spaces, Studia Mathematica, XXXV (1970) 273–292.

[4] Boman, J., Partial regularity of mappings between Euclidean spaces. Acta Math. 119. 1967. 1–25.

[5] Hormander, L., *The analysis of linear partial differential operators I*, Springer-Verlag.

[6] Korevaar, J., Applications of $\mathbb{C}^n$ capacities. Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), 105–118,

[7] Krantz S. G. and Parks H.R., *A primer of real analytic functions*. Second edtion. Birkhauser, 2002.

[8] Neelon, T. S., On separate ultradifferentiability of functions. Acta Sci. Math. (Szeged) 64 (1998) 489–494.

[9] Neelon, T. S., Ultradifferentiable functions on lines in $\mathbb{R}^n$. Proc. Amer. Math. Soc. 127 (1999) 2099–2104.

[10] Neelon, T. S., A Bernstein–Walsh type inequality and applications. Canad. Math. Bull. 49 (2006) 256–264.

[11] Neelon, T. S., Restrictions of power series and functions to algebraic surfaces. Analysis, Vol. 29, Issue 1 (2009), page 1-15.

[12] Rudin, W., *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill. 1976.

[13] Siciak, J.: A characterization of analytic functions of n real variables, Studia Mathematica, XXXV (1970) 293–297.

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